Topology Term Paper: A Brief Introduction to Conformal Field Theory

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Contents

Introduction

This term paper is a lightning introduction to conformal field theory (CFT). I will try to lay out the bare minimum amount of theory to derive and compute the conformal blocks of the Ising CFT in order to derive its anyonic excitations. It is almost entirely from the book by Di Francesco et. al. I changed some derivations to be more brief and to the point, and filled in the details “left to the reader” in other parts. I cut out all notions of Lorentz invariance and instead just start from locally conformal functions. I changed the order of presentation of some concepts in a way I found more appealing.

Rudiments of Conformal Field Theory

Every book on complex analysis tells us conformal transformation is a function $f : \mathbb{C} \mapsto \mathbb{C}$ that preserves angles. Recalling a result from complex analysis, we know that the conformal transformations are exactly the locally analytic and one-to-one functions. Hence we can expand such functions in a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} \xi_n z^n$$

with $\xi_n \in \mathbb{C}$. The global conformal transformations (those conformal transformations that are everywhere analytic) are the three parameter group of linear fractional transformations

$$g(z) = \frac{az+b}{cz+d}; \text{ with } ad - bc = 1$$

In the terminology of conformal field theory these form the special conformal group under the multiplication operation implied by identification with matrices (the group is $SL_2(\mathbb{C})/\mathbb{Z}_2$). For special values of the parameters a, b, c, and d, we have:

1. Translations: $a = 1, c = 0, d = 1 \Rightarrow x \to x + b$
2. Dilations: $b = 0, c = 0, d = 1 \Rightarrow x \to ax$
3. Special Conformal: $a = 1, b = 0, d = 1 \Rightarrow x \to x/(cx + 1)$

Conformal Transformations of Functions

First we construct the Lie algebra for the group of conformal transformations of functions. Consider the following infinitesimal conformal transformation

$$z \to w(z) = z + \epsilon\xi(z) = z + \epsilon \sum_{n=\infty}^{n=\infty} \xi_n z^{n+1}$$

Functions then transform as

$$\lim_{\epsilon \to 0} \frac{f(z + \epsilon\xi(z)) - f(z)}{\epsilon} = \frac{df}{d\epsilon}|_{\epsilon=0} = -\xi(z)\frac{\partial f}{\partial z}$$

We can then define the generators of the transformation as

$$G_\xi = -i\xi(z)\frac{\partial}{\partial z} = -i \sum_{n=-\infty}^{n=\infty} \xi_n z^{n+1} \frac{\partial}{\partial z} \equiv i \sum_{-\infty}^{\infty} \xi_n L_n$$

where $L_n = -z^{n+1}\frac{\partial}{\partial z}$ is a basis for the Lie algebra. This calculation shows that the Lie algebra is infinite dimensional, which is obvious because there are an infinite number of coefficients for a holomorphic function. Of course a Lie algebra needs a bracket:

$$
\begin{aligned}
[L_m, L_n] &= -z^{m+1}\frac{\partial}{\partial z}\left[z^{n+1}\frac{\partial}{\partial z}\right] + z^{n+1}\frac{\partial}{\partial z}\left[z^{m+1}\frac{\partial}{\partial z}\right] \\
&= -z^{m+1}(n+1)z^n\frac{\partial}{\partial z} - z^{m+n+2}\frac{\partial^2}{\partial z^2} + z^{n+1}(m+1)z^m\frac{\partial}{\partial z} + z^{n+m+2}\frac{\partial^2}{\partial z^2} \\
&= (m-n)z^{m+n+1}\frac{\partial}{\partial z} \\
&= (m-n)L_{m+n}
\end{aligned}
$$

This is the Witt algebra, originally due to Cartan, the algebra of derivations over the Laurent polynomials. In passing to the transformations of quantum fields, we will need extend the Witt algebra by a one-dimensional abelian Lie algebra, resulting in the Virasoro algebra. The central extension is a pretty common tool in passing from classical to quantum theories.

Basic Manipulations of Conformal Fields

We study quantum fields on Euclidean space. Define the holomorphic conformal dimension h and the anti-holomorphic conformal dimension $\bar{h}$. A field ϕ is called *primary* if it transforms in the following way under the conformal maps $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$:

$$
\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z, \bar{z}) \tag{1}
$$

Fields that are not primary are *secondary.* A field may also be *quasi-primary* if it transforms as equation (1) under *global* conformal transformations. Therefore, a quasi-primary field may either be secondary or primary. An important example of a secondary field is the energy-momentum tensor, defined by the relation

$$
\delta S = \int d^2x \partial_\mu T^{\mu\nu}\epsilon_\nu
$$

under the infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu$. Conformal invariance places strong constraints on the observables (correlation functions) of the theory. Let us calculate the 2- and 3-point correlation functions under global conformal transformations. We have by definition

$$
\begin{aligned}
&\langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)\rangle \\
&\qquad = \left(\frac{dw_1}{dz}\right)^{-h_1}\left(\frac{d\bar{w}_1}{d\bar{z}}\right)^{-\bar{h}_1}\left(\frac{dw_2}{dz}\right)^{-h_2}\left(\frac{d\bar{w}_2}{d\bar{z}}\right)^{-\bar{h}_2}\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle
\end{aligned}
$$

Under scale transformations $z = \lambda w$ we get

$$\langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)\rangle = \lambda^{-h_1-\bar{h}_1-h_2-\bar{h}_2}\langle \phi_1(\lambda w_1, \lambda\bar{w}_1)\phi_2(\lambda w_2, \lambda\bar{w}_2)\rangle$$

Invariance under translation and rotations imply that $\langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)\rangle = f(|w_1 - w_2|, |\bar{w}_1 - \bar{w}_2|)$. The scale transformations imply that f is a homogeneous function in both its arguments of of degree $-h_1 - \bar{h}_1$ and $-h_2 - \bar{h}_2$ resepectively. This constrains the 2-point correlation function to be of the form

$$\langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)\rangle = \frac{C_{12}}{|w_1 - w_2|^{h_1+h_2}|\bar{w}_1 - \bar{w}_2|^{\bar{h}_1+\bar{h}_2}}$$

Lastly, the special conformal transformations $w' = 1/(cw + d)$ have Jacobian $|\partial w'/\partial w| = -c/(cw + d)^2 = -cw'^2$. Distances between points have the transformation rule

$$|1/(cw_1 + d) - 1/(cw_2 + d)| = cw_1'w_2'|w_1 - w_2|$$

so that

$$\frac{C_{12}}{|w_1 - w_2|^{h_1+h_2}|\bar{w}_1 - \bar{w}_2|^{\bar{h}_1+\bar{h}_2}} =$$
$$[-cw_1'^2]^{-h_1}[-cw_2'^2]^{-h_2}[-c\bar{w}_1'{}^2]^{-\bar{h}_1}[-c\bar{w}_2'^2]^{-\bar{h}_2}\frac{[cw_1'w_2']^{h_1+h_2}[c\bar{w}_1'\bar{w}_2']^{\bar{h}_1+\bar{h}_2}}{|w_1 - w_2|^{h_1+h_2}|\bar{w}_1 - \bar{w}_2|^{\bar{h}_1+\bar{h}_2}}$$

For this relation to hold we need $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$. Therefore the correlation function is

$$\langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)\rangle = \begin{cases} \frac{C_{12}}{|w_1-w_2|^{2h}|\bar{w}_1-\bar{w}_2|^{2\bar{h}}} & \text{whenever } h_1 = h_2 = h, \bar{h}_1 = \bar{h}_2 = \bar{h} \\ 0 \text{ otherwise.} \end{cases}$$

For the 3-point function

$$\langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)\phi_3(w_3, \bar{w}_3)\rangle = C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_3+h_1-h_2}} \times \text{anti-holomorphic}$$

where the antiholomorphic part is the exact same expression but with bars over every term. Also, $z_{12} = |z_1 - z_2|$, etc.

The conformal Ward identities express the conservation laws derived from invariance under the continuous conformal symmetry. (They are just specific statements of Noether's Theorem). Actually they are only valid in the sense of distributions, i.e. they are meaningless unless integrated. Let us describe the general conformal Ward identity. We are interested in relations constraining the correlation function of the string of primary operators $X = \phi(x_1)\cdots\phi(x_n)$

$$\langle X\rangle = \frac{1}{Z}\int [d\phi]\exp(-S[\phi])\phi(x_1)\cdots\phi(x_n)$$

Consider the the arbitrary conformal coordinate variation $\epsilon^\nu(x)$. Effecting the variation (and being extremely sketchy) and substituting in the definition of the energy-momentum tensor:

$$\begin{aligned}
\delta_\epsilon\langle X\rangle &= \frac{1}{Z}\int [d\phi]\exp(-S[\phi])\delta_\epsilon S\phi(x_1)\cdots\phi(x_n)\\
&= \frac{1}{Z}\int [d\phi]\exp(-S[\phi])\int d^2x\partial_\mu T^{\mu\nu}\epsilon_\nu\phi(x_1)\cdots\phi(x_n)\\
&= \int d^2x\partial_\mu\langle T^{\mu\nu}\epsilon_\nu X\rangle
\end{aligned}$$

We can identify the different components of the variation by doing the derivative:

$$\begin{aligned}
\partial_\mu(\epsilon_\nu T^{\mu\nu}) &= \epsilon_\nu\partial_\mu T^{\mu\nu} + (\partial_\mu\epsilon_\nu)T^{\mu\nu}\\
&= \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu)T^{\mu\nu}\\
&= \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho\epsilon^\rho\eta_{\mu\nu})T^{\mu\nu} + \frac{1}{2}(\epsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\epsilon_{\mu\nu})T^{\mu\nu}
\end{aligned}$$

where we broke up the derivative $\partial_\mu\epsilon_\nu$ into its symmetric and antisymmetric parts and rewrote them in terms of the symmetric metric tensor and antisymmetric Levi-Civita symbol. The first term is translations, the second the local scaling factor, and the third term is local rotations. This form is not really useful. What is useful is the expression for the holomorphic part of the correlation function

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left(\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right) + \text{reg.} \tag{2}$$

where w_i is the argument of the ith field in X and h_i its conformal dimension. Also, reg. stands for an arbitrary holomorphic (analytic, regular) function.

The last basic CFT concept is the operator product expansion, which expands operators inside a correlation function in terms of the other operators in the theory. In particular, whenever two operators, functions of z and w, are considered in a correlation function, the OPE gives the asymptotic expression of the correlation function in the limit $z \to w$. For example, the holomorphic Ward identity (2) with $X = \phi(w, \bar{w})$ leads to the OPE

$$T(z)\phi(w,\bar{w}) \sim \frac{1}{z-w}\partial_w \phi(w,\bar{w}) + \frac{h}{(z-w)^2}\phi(w,\bar{w})$$

The OPE includes terms that are singular as $z \to w$. In general for an arbitrary string of operators O_i:

$$\langle O_i(z,\bar{z})O_j(w,\bar{w})\cdots\rangle = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\langle O_k(w,\bar{w})\cdots\rangle$$

A crucial result is the OPE of the energy momentum tensor:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

where c is the central charge. We will derive this expression for the free fermion in order to get its central charge.

Quantization: Definition of the Hilbert Space

To understand the Hilbert space structure, we need to define the radial quantization. I will just sketch the general idea. If we are working in a 2 dimensional Euclidean space, we pick a direction to be the time and one to be the space. The space direction is compactified into a circle so that the whole space is a cylinder. Then the cylinder is mapped onto the complex plane such that the space direction is mapped to angles and the time is mapped to the radial coordinate. We need not formulate our theories on the complex plane, and in fact they can be calculated on any Riemann surface, such as the torus or a surface of higher genus.

Now we assume the existence of the vaccuum state $|0\rangle$ and define the asymptotic state as $|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})\,|0\rangle$. We define Hermitian conjugation on the real surface of the quasi-primary field operator with conformal dimensions $(h,\bar{h})$ as $[\phi(z,\bar{z})]^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\phi(1/\bar{z},1/z)$. The asymptotic out state is of course $|\phi_{\text{out}}\rangle = [|\phi_{\text{in}}\rangle]^\dagger$. We can also define the mode expansion of the field operators:

$$\phi(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n}$$

Using orthogonality of the Laurent polynomials, we can invert the transformation as:

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+\bar{h}-1}\phi(z,\bar{z})$$

where the contour integral circles the origin. It is customary to drop the antiholorphic parts. In this case the mode expansion is written

$$\phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} z^{-m-h} \phi_m$$

$$\phi_m = \frac{1}{2\pi i} \oint dz z^{m+h-1} \phi(z)$$

Lastly, time-ordering is converted to radial ordering under the map described above. We write

$$\mathcal{R}\phi_1(z)\phi_2(w) = \begin{cases} \phi_1(z)\phi_2(w) \text{ if } |z| > |w| \\ \pm\phi_2(w)\phi_1(z) \text{ if } |w| > |z| \end{cases}$$

where the $\pm$ is for bosons and fermions. After some algebra it is possible to show that for two operators

$$A = \oint a(z)dz \text{ and } B = \oint b(z)dz$$

their commutation relation is

$$[A, B] = \oint_0 dw \oint_w dz a(z) b(w)$$

This can be shown by the residue theorem. So commutation relations of operators are given concrete meaning in terms of contour integrals. Also, the mathematical meaning of the operator product expansion is now obvious: clearly only singular parts of complex functions contribute to evaluation of residues. The holomorphic parts are irrelevant.

Operators: The Virasoro Algebra

Next we need to describe the Virasoro algebra and the Verma modules that allow for classification of simple conformal field theories. Analagous to the Lie algebra of conformal transformations derived above, we need to construct the Lie algebra of conformal transformations over the Hilbert space of the CFT. This is the Virasoro algebra. First define the conformal charge

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z)$$

which is the generator of conformal transformations. Then insert the OPE of energy-momentum tensor into the mode expansion defined by

$$T(z) \equiv \sum_{n \in \mathbb{Z}} z^{-n-2} L_n; L_n \equiv \frac{1}{2\pi i} \oint dz z^{n+1} T(z)$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n; \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+1}\bar{T}(\bar{z})$$

as well as the infinitesimal conformal charge:

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n$$

Then

$$\begin{aligned}
Q_\epsilon &= \frac{1}{2\pi i}\oint dz\epsilon(z)T(z) \\
&= \frac{1}{2\pi i}\oint dz \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n \sum_{m\in\mathbb{Z}} z^{-m-2}L_m \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \epsilon_n L_m \frac{1}{2\pi i}\oint dz z^{n-m-1} \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \epsilon_n L_m \delta_{nm} \\
&= \sum_{n\in\mathbb{Z}} \epsilon_n L_n
\end{aligned}$$

Therefore the mode operators L_n and $\bar{L}_n$ are the generators of local conformal transformations (almost by definition). Now let us compute their commutators, which will make use of the definition of the commutation relation and the operator product expansion above

$$\begin{aligned}
[L_n, L_m] &= \left(\frac{1}{2\pi i}\right)^2 \oint dw w^{m+1} \oint dz z^{n+1} \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\
&= \frac{1}{2\pi i}\oint dw w^{m+1}\left((n+1)(n)(n-1)w^{n-2}(c/2)/(3!) + 2T(w)w^n(n+1)/(1!) + w^{n+1}\partial T(w)\right) \\
&= \frac{1}{2\pi i}\oint dw (n+1)(n)(n-1)w^{n+m-1}c/12 + 2T(w)w^{n+m+1}(n+1) + w^{n+m+2}\partial T(w) \\
&= (n+1)(n)(n-1)c/12\delta_{n+m,0} + 2(n+1)L_{m+n} + \frac{1}{2\pi i}\oint dw w^{n+m+2}\partial T(w) \\
&= (n+1)(n)(n-1)c/12\delta_{n+m,0} + 2(n+1)L_{m+n} - \frac{1}{2\pi i}\oint dw (n+m+2)w^{n+m+1}T(w) \\
&= (n+1)(n)(n-1)c/12\delta_{n+m,0} + 2(n+1)L_{m+n} - (n+m+1)L_{n+m} \\
&= \frac{1}{12}cn(n^2-1)\delta_{n+m,0} + (n-m)L_{m+n}
\end{aligned}$$

The end result is that the generators $L_n, \bar{L}_n$ obey the *Virasoro algebra*:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$
$$[L_n, \bar{L}_m] = 0$$
$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

c is the central charge of the theory. It has an interpretation in terms of anomalies. The operators L_0 and $\bar{L}_0$ describe dilations and hence $L_0 + \bar{L}_0$ is proportional to the Hamiltonian in radial quantization. Now that we have the algebra of operators we can construct a description of the Hilbert space. The vacuum is defined as

$$L_n |0\rangle = 0; \bar{L}_n |0\rangle = 0, n \geq -1$$

which expresses invariance under global conformal transformations. Now we can relate the action of the primary fields to the action of the operators. The operators $\phi(z, \bar{z})$ with conformal dimension $(h, \bar{h})$ act on the vaccuum to produce the asymptotic states $|h, \bar{h}\rangle \equiv \phi(z, \bar{z})|0\rangle$. Let us now specialize to the holomorphic part by dropping all references to $\bar{h}$ and anti-holomorphic transformations, which we can do because the Virasoro algebra factorizes into the two sectors of operators.

We can describe a family of states that transform among themselves under conformal transformations (thus furnishing a representation of the Virasoro algebra). Such states are the *descendent* states of the state $|h\rangle$, given by $L_{-n}\phi(0)|0\rangle$ (in general strings of L_i). The corresponding field that creates this state is $\phi^{-n}(w) \equiv L_{-n}\phi(w)$. Let $X = \phi_1(w_1)\cdots\phi_N(w_N)$ be a string of primary fields. Then the correlator can be calculated by inserting the definition of L_{-n} in terms of $T(z)$ and then using the OPE derived from the conformal Ward identity:

$$\begin{aligned}
\langle\phi^{(-n)}(w)X\rangle &= \langle(L_{-n}\phi(w))X\rangle \\
&= \langle(\frac{1}{2\pi i}\oint dz(z-w)^{-n+1}T(z)\phi(w))X\rangle \\
&= \frac{1}{2\pi i}\oint_w dz(z-w)^{-n+1}\langle T(z)\phi(w)X\rangle \\
&= \frac{1}{2\pi i}\oint_w dz(z-w)^{-n+1}\sum_i\left(\frac{1}{z-w_i}\partial_{w_i}\langle\phi(w)X\rangle + \frac{h_i}{(z-w_i)^2}\langle\phi(w)X\rangle\right) \\
&= -\frac{1}{2\pi i}\oint_{w_i} dz(z-w)^{-n+1}\sum_i\left(\frac{1}{z-w_i}\partial_{w_i}\langle\phi(w)X\rangle + \frac{h_i}{(z-w_i)^2}\langle\phi(w)X\rangle\right)
\end{aligned}$$

In the last step, the contour in the definition of L_{-n} circling w was reversed to surround every other pole w_i, flipping the sign. By the residue theorem,

$$\begin{aligned}\langle \phi^{(-n)}(w)X\rangle &= -\frac{1}{2\pi i}\sum_i \left((w_i - w)^{-n+1}\partial_{w_i}\langle\phi(w)X\rangle + \frac{(-n+1)h_i}{(w_i-w)^{-n}}\langle\phi(w)X\rangle\right)\\ &= \underbrace{\frac{1}{2\pi i}\sum_i \left(-(w_i - w)^{-n+1}\partial_{w_i} + \frac{(n-1)h_i}{(w_i-w)^{-n}}\right)}_{\mathcal{L}_{-n}}\langle\phi(w)X\rangle\end{aligned}$$

which defines a differential operator $\mathcal{L}_{-n}$ and a differential equation that relates the correlator of the primary fields with the descendent fields. A primary field together with its descendent fields comprise a conformal family. The point is that the correlators of the descendent fields are determined by the primary fields and the energy-momentum tensor. The last curcial concept we need is that the descendant states have eigenvalues completely determined by the holomorphic dimension of the ancestor state and the central charge. This can be verified by the commutation relations. The eigenvalues corresponding to one family of states are called a *conformal tower*.

Representations: Verma Modules

The Hilbert space of states can be divided into the representations of the Virasoro algebra, derived above. Now we describe the highest-weight representations. We pick a distinguished generator L_0 and a ket $|h\rangle$ as the highest-weight state such that $L_0\,|h\rangle = h\,|h\rangle$. Now we note that $[L_0, L_m] = -mL_m$ for $m > 0$ which implies that L_m is a lowering operator and L_{-m} is a raising operator. (This is confusing terminology. We could have called it a lowest-weight representation and swapped the names of the raising and lower operators.) We require $L_m\,|h\rangle = 0$ with $m > 0$. The other states in the representation can be applied by acting successively with L_{-k_i}:

$$|h'\rangle = L_{-k_1}L_{-k_2}\cdots L_{-k_n}\,|h\rangle(1 \le k_1 \le \cdots \le k_n)$$

Now we know that $h' = h + k_1 + k_2 + \cdots + k_n = h + N$. N is the level of the state. The level of the string of operators is $h + N$. Clearly the number of unique kets at level N is the number of ways to partition the integer N. This will be important in a little bit. The adjoints of the generators are $L_m^\dagger = L_{-m}$. This defines an inner product in the usual way. A representation with highest weight h of the holomorphic generators L_k associated with a Virasoro algebra with central charge c will be called the Verma module $V(c, h)$. As usual every representation $V(c, h)$ has an invariant character $\chi_{(c,h)}(\tau)$ defined by the trace:

$$\chi_(c,h)(\tau) = \operatorname{Tr} q^{L_0 - c/24} = \sum_n^\infty \dim(h+n)q^{n+h-c/24}$$

for which the motivation of the phase factors is beyond the scope of this paper. Now in fact the Verma Modules may or may not be irreducible representations. There is an easy way to check: if the null space of the operators is degenerate then the module can't be an irreducible representation. Now we describe the so-called minimal models. If we specify that the descendent fields of a given operator are null vectors at a certain level, it follows that all descendents further than those will also be null. Then the number of non-trivial operators is finite, and the CFT is greatly simplified. The derivation of this fact is beyond the scope of what can go in this term paper. I will only give a vague outline. We consider the Gram matrix $M = \langle i|j\rangle$ of inner products between basis states. Basis states corresponding to different levels are orthogonal, hence the Gram matrix is block diagonal with block matrices $M^{(l)}$. A model is said to be unitary if M is positive definite. M contains null vectors if there are zero eigenvectors. So, we can check if we have zero eigenvectors by computing the determinant of each block $M^(l)$. The data for a CFT are the conformal charge c and holomorphic dimension h, so M will have entries that are functions of c and h. Hence the eigenvalues will be functions of c and h. In principle, we can calculate the eigenvalues which we label $h_{r,s}$ where $rs \leq l$ for a given block $M^{(l)}$. The Kac determinant formula

$$\det M^{(l)} = \alpha_l \prod_{r,s\geq 1, rs\leq l} (h - h_{r,s}(c))^{\mathrm{part}(l-rs)}$$

where

$$h_{r,s}(t) = \frac{1}{4}(r^2-1)t + \frac{1}{4}(s^2-1)\frac{1}{t} - \frac{1}{2}(rs-1)$$
$$c = 13 - 6(t + 1/t)$$

gives the general formula for the determinant of the blocks of the Gram matrix, where t is a complex parameter that lets us walk through different values of the central charge (just to simplify the expressions). The part stands for partitions, and is the number of ways of partitioning the integer $l - rs$. Now we observe the following fact. If $h = h_{r,s}$, then the Kac determinant is zero. Hence there is a null vector at level $l = rs$. Therefore, the operator algebra is truncated as described above. Such a truncation places a restriction on which three point functions vanish, in addition to the constraints described in an earlier section. The trajectories of the eigenvalues have been studied, and the parameter regimes of c and h which give valid theories have been mapped out. These are the so-called minimal models.

What is it good for?

We have developed a lot of theory in the past few sections concerning conformal field theory in general. But how can I, a condensed matter theorist, use it in my research? The procedure is to go to DMRG or ED and calculate the eigenvalues for a given 1+1d quantum system near criticality. Then, fit one or more conformal towers to the eigenvalues, which extracts the conformal charge and scaling dimensions of the conformal field theory that describes the transition. Then, study the conformal field theory which allows scaling to the infinite volume limit and for large systems finite-size corrections can be added. The central charge can also be extracted from the ground state, from which some thermodynamic quantities can be calculated. So in principle Lanczos or some inexact methods can be used. Conformal field theory is also useful for calculation of the entanglement entropy.

Some Results for the Ising CFT

Here I list some results for the Ising CFT, the simplest minimal model, which demonstrate some further concepts in conformal field theory. First, we introduce the XZ model:

$$\hat{H} = -\sum_j \hat{\sigma}_j(n) + -g\sum_j \hat{\sigma}^z_j \hat{\sigma}^z_{j+1}$$

Which is equivalent at criticality to the free electron field theory with action

$$\mathcal{S} = g\int \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}$$

In terms of the free electron operators the holomorphic part of the energy momentum tensor is

$$T^{zz} = 2g\bar{\psi}\bar{\partial}\bar{\psi}$$

and the antiholomorphic part is

$$T^{\bar{z}\bar{z}} = 2g\psi\bar{\partial}\psi$$

The correspondence is derived by the Jordan-Wigner transformation and the continuum limit. It can be shown that this model is equivalent to the $M(4,3)$ minimal by direct calculation of the energy-momentum tensor and its operator product expansion by the usual methods of field theory. In the $M(4,3)$ minimal model there are three primary fields $I = \phi_(1,1)$, $\sigma = \phi_(2,2)$ $(h = \bar{h} = 1/16)$, and $\varepsilon = \phi_(2,1)$ $(h = \bar{h} = 1/2)$. The central charge is $c = 1/2$. The $M(4,3)$ is equivalent to the free electron gas with operators ψ and $\bar{\psi}$. The energy density is related to these fields by $\psi\bar{\psi}$ but the relation to the primary field σ is highly nonlocal, and is related by the Jordan-Wigner transformation to the operators of the free electron gas. I list an OPE we will use later:

$$\psi(z)\sigma(w,\bar{w}) \sim \frac{1}{(z-w)^{1/2}}\mu(w,\bar{w})$$

where μ is the disorder operator dual to the local energy operator ε.

$p_x + ip_y$ Superconductors and Chiral Edge Modes

This section is based on the lecture notes by J Dubail: CFT and 2d chiral topological phases: an introduction. As before, I fill in the details in some places (the so-called simple calculations) and cut them in others.

Here we come to the second application, which is the use of CFTs to construct variational states for studying topological phases, such as in the Moore-Read wavefunctions for the fractional quantum hall effect. Here I will describe how the Ising CFT suggests variational states for the $p_x + ip_y$ superconductor. Such a superconductor is described by a mean-field theory that captures the transition from the trivial phase (s-wave) at values of the chemical potential $\mu < 0$ to the topological phase at $p_x + ip_y$. The mean-field Hamiltonian matrix is:

$$H(k) = \begin{pmatrix} \frac{k^2}{2m} - \mu & \Delta(k) \\ \Delta^*(k) & -\frac{k^2}{2m} - \mu \end{pmatrix}$$

Here $\Delta(k)$ is the gap function. Now we want to examine the topological properties of this Hamiltonian. In this course we learned to do this essentially by examining the characteristic class of the eigen-vector bundle induced by the map $H(k)$. Here we will take a different approach, in which we deform the Hamiltonian by taking an appropriate limit into one that is describable by a conformal field theory. In fact CFTs and topological field theories (which are constructed by the gauge-invariant Wilson loops) are very closely related. But I do not really understand this correspondence so I will not elaborate on it. Back to the problem at hand: such a deformation is the long wavelength limit: these superconducting states are defined by the short range properties of the gap function $\Delta(k) \sim k_x - ik_y$. BCS theory asserts that the coherent state

$$|\Psi\rangle = \exp\left(\frac{1}{2}\int \frac{d^2k}{(2\pi)^2} g_k c^\dagger_{-k} c^\dagger_k\right)|0\rangle$$

is the ground state of the superconductor, where g_k is given by

$$g_k \equiv \frac{\frac{k^2}{2m} - \mu - \sqrt{(\frac{k^2}{2m} - \mu)^2 + |\Delta(k)|^2}}{\Delta^*(k)}$$

For finite μ The leading behavior of g_k for small k is given by the singular contribution from the denominator:

$$g_k \sim \frac{1}{k_x + ik_y}$$

Now we formulate a trial wavefunction in real space. We have

$$
\begin{aligned}
g(x,y) &= \int \frac{d^2k}{(2\pi)^2} e^{i(k_x x + k_y y)} \frac{1}{k_x + ik_y} \\
&= \int \frac{dk_y}{2\pi} e^{ik_y y} \int \frac{dk_x}{2\pi} e^{ik_x x} \frac{1}{k_x + ik_y} \\
&= \int \frac{dk_y}{2\pi} e^{ik_y y + k_y x} \\
&= \frac{\kappa}{x + iy}
\end{aligned}
$$

where κ is a constant and we assume suitable regularization of the Fourier transform. Note that accordingly this approximation for g is in the long distance $|x + iy| \to \infty$ limit. Now we manipulate the BCS wavefunction:

$$
\begin{aligned}
|\Psi\rangle &= \exp\left(\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k_x + ik_y} \int d^2r e^{-ikr} c_r^\dagger \int d^2r' e^{ikr'} c_{r'}^\dagger \right) |0\rangle \\
&= \exp\left(\frac{1}{2} \int d^2r \int d^2r' c_r^\dagger c_{r'}^\dagger \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik(r-r')}}{k_x + ik_y} \right) |0\rangle \\
&\sim \exp\left(\frac{1}{2} \int d^2r \int d^2r' \frac{\kappa}{(x - x') + i(y - y')} c_r^\dagger c_{r'}^\dagger \right) |0\rangle \\
&= \exp\left(\frac{\kappa}{2} \int dz_1 \int dz_2 \frac{1}{z_1 - z_2} c_{z_1}^\dagger c_{z_2}^\dagger \right) |0\rangle
\end{aligned}
$$

where in the last step I make the complexification $r \to z_1$, $r' \to z_2$. We can now take this as a trial wavefunction to describe the $\mu > 0$ region in the long wavelength limit or the large separation limit. In fact, it is exact in the limit. Now we can start using our results from CFT. We introduce the new field ψ with conformal dimension 1/2, the field of the Ising CFT. The correlator of the Ising CFT is:

$$
\langle \psi(z_1) \psi(z_2) \rangle = \frac{1}{z_1 - z_2}
$$

which we can insert into the trial wavefunction:

$$
\begin{aligned}
|\Psi'\rangle &= \exp\left(\frac{\kappa}{2}\int dz_1 \int dz_2 \langle\psi(z_1)\psi(z_2)\rangle c^\dagger_{z_1} c^\dagger_{z_2}\right)|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\kappa}{2}\int dz_1 \int dz_2 \langle\psi(z_1)\psi(z_2)\rangle c^\dagger_{z_1} c^\dagger_{z_2}\right]^n |0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\frac{\kappa}{2}\int dz_1 \int dz_2 \psi(z_1)\psi(z_2) c^\dagger_{z_1} c^\dagger_{z_2}\right\rangle^n |0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\frac{-\kappa}{2}\int dz_1 \int dz_2 \psi(z_1) c^\dagger_{z_1}\psi(z_2) c^\dagger_{z_2}\right\rangle^n |0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\frac{i\sqrt{\kappa}}{\sqrt{2}}\int dz_1 \psi(z_1) c^\dagger_{z_1}\frac{i\sqrt{\kappa}}{\sqrt{2}}\int dz_2 \psi(z_2) c^\dagger_{z_2}\right\rangle^n |0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{2}{n!}\left\langle\frac{i\sqrt{\kappa}}{\sqrt{2}}\int dz_1 \psi(z_1) c^\dagger_{z_1}\right\rangle^{2n} |0\rangle \\
&= \left\langle\exp\left(i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})\right)\right\rangle|0\rangle
\end{aligned}
$$

So we have deformed the wavefunction into one described by the composite quasiparticle with creation operator $\psi c^\dagger_k$. In the fractional quantum hall effect, an analagous construction for the Laughlin wavefunctions produces the coherent state of charge-flux composites. The adjoint vector is

$$
\langle\Psi'| = \langle 0|\left\langle\exp\left((\kappa*)^{1/2}\int d^2z c(z,\bar{z})\bar{\psi}(\bar{z})\right)\right\rangle
$$

Being careful to note that there is no factor of i in the exponential. Now we note the following fact. Acting with $c(w,\bar{w})$ on $|\Psi\rangle$ gives

$$
\begin{aligned}
&c(w,\bar{w})\left\langle\exp\left(i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})\right)\right\rangle|0\rangle \\
&= \left\langle\sum_{n=1}^{\infty}\frac{1}{n!}[-i\kappa^{1/2}\int d^2z\psi(z)(\delta_{w,z}\delta_{\bar{w},\bar{z}} - c^\dagger(z,\bar{z})c(w,\bar{w}))][i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})]^{n-1}\right\rangle|0\rangle \\
&= \cdots \text{commute } n-1 \text{ more times} \cdots \\
&= (-i\kappa^{1/2}\psi(w))\left\langle\sum_{n=1}^{\infty}\frac{1}{(n-1)!}[i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})]^{n-1}\right\rangle|0\rangle \\
&= (-i\kappa^{1/2}\psi(w))\left\langle\exp\left(i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})\right)\right\rangle|0\rangle
\end{aligned}
$$

We also need the fact that

$$\begin{aligned}
&\langle 0|\left\langle \exp\left((\kappa*)^{1/2}\int d^2w c(w,\bar{w})\bar{\psi}(\bar{w})\right)\right\rangle \left\langle \exp\left(i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})\right)\right\rangle |0\rangle \\
&= \langle 0|\sum_{mn}\frac{1}{n!m!}\left\langle (\kappa*)^{1/2}\int d^2w c(w,\bar{w})\bar{\psi}(\bar{w})\right\rangle^m \left\langle i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})\right\rangle^n |0\rangle \\
&= \sum_m \frac{1}{m!} i|\kappa|^m \left\langle \int d^2z\bar{\psi}(\bar{z})\psi(z)\right\rangle^m \text{There are } n! \text{ ways to annihilate } c(\bar{w}_i). \\
&= \left\langle \exp\left(|\kappa|\int d^2z i\bar{\psi}(\bar{z})\psi(z)\right)\right\rangle \\
&= \left\langle \exp\left(|\kappa|\int d^2z\epsilon(z,\bar{z})\right)\right\rangle
\end{aligned}$$

Where in the last line we use (define?) the energy operator $\epsilon(z,\bar{z}) = i\bar{\psi}(\bar{z})\psi(z)$. Now we have everything we need to start computing correlation functions for the superconductor. The procedure will be to convert the correlation functions $\langle\Psi| c(z)c^\dagger(w)|\Psi\rangle$ to correlators $\langle\Psi|\bar{\psi}(\bar{w})\psi(w)|\Psi\rangle$. These can be efficiently evaluated by conformal field theory, as we will see: we will do an example for the boundary of a $p_x + ip_y$ superconductor. We take the upper half plane $\mathbb{H}$ as belonging to the trivial phase (the vacuum) and the lower half plane $\mathbb{C}\setminus\mathbb{H}$ belonging to the superconductor. Then the real axis is the boundary of the two phases. On the boundary we impose the conformal boundary condition $T(z) = \bar{T}(\bar{z}) \Rightarrow \bar{\psi}(\bar{z}) = \psi(z)$ (see above). The first equation expresses the condition that there be no momentum flow across the boundary. We wish to calculate correlators of the original fermionic fields on the real axis, i.e.

$$\begin{aligned}
&\langle c^\dagger(x,x)c(y,y)\rangle \\
&= \frac{\langle\Psi'| c^\dagger(x,x)c(y,y)|\Psi'\rangle}{\langle\Psi'|\Psi'\rangle} \\
&= \kappa\frac{\langle\Psi'|\bar{\psi}(x)\psi(y)|\Psi'\rangle}{\langle\Psi'|\Psi'\rangle} \\
&= \kappa\langle 0|\left\langle \exp^{(\kappa^*)^{1/2}\int_{\mathbb{C}\setminus\mathbb{H}} d^2zc(z,\bar{z})\bar{\psi}(\bar{z})}\right\rangle \bar{\psi}(x)\psi(y)\left\langle e^{i\kappa^{1/2}\int_{\mathbb{C}\setminus\mathbb{H}} d^2z\psi(z)c^\dagger(z,\bar{z})}\right\rangle |0\rangle \,/\, \langle\Psi'|\Psi'\rangle \\
&= \kappa\langle 0|\left\langle \bar{\psi}(x)\psi(y)\exp^{(\kappa^*)^{1/2}\int_{\mathbb{C}\setminus\mathbb{H}} d^2zc(z,\bar{z})\bar{\psi}(\bar{z})} e^{i\kappa^{1/2}\int_{\mathbb{C}\setminus\mathbb{H}} d^2z\psi(z)c^\dagger(z,\bar{z})}\right\rangle |0\rangle \,/\, \langle\Psi'|\Psi'\rangle \\
&= \left\langle \bar{\psi}(x)\psi(y)\exp\left(|\kappa|\int_{\mathbb{C}\setminus\mathbb{H}} d^2z\epsilon(z,\bar{z})\right)\right\rangle \,/\, \langle\Psi'|\Psi'\rangle \\
&= \frac{\left\langle \bar{\psi}(x)\psi(y)\exp\left(|\kappa|\int_{\mathbb{C}\setminus\mathbb{H}} d^2z\epsilon(z,\bar{z})\right)\right\rangle}{\left\langle \exp\left(|\kappa|\int d^2z\epsilon(z,\bar{z})\right)\right\rangle}
\end{aligned}$$

We recognize this as the correlation functions in an Ising CFT plus a mass term, i.e. a CFT perturbed from criticality. Such perturbations attenuate the power law correlations to exponential correlations which we expect away from the gapless critical point. If we impose the conformal boundary condition $\bar{\psi}(\bar{z}) = \psi(z)$ then we get a chiral theory, i.e. one in which we only have holomorphic fields. Therefore we have derived a chiral boundary conformal field theory for the interface between the $p_x + ip_y$ superconductor and the vacuum. The presence of a chiral edge mode is an indicator of a topological phase. The boundary CFT can of course be studied further to extract information about the bulk.

Phase winding around vortices in the superconductor can also be studied. The presence of vortices is added in by multiplying the trial wave function by the vortex operators σ. For vortices at η_1 and η_2 the trial wavefunction is

$$\left\langle \sigma(\eta_1)\sigma(\eta_2)\exp\left(i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})\right)\right\rangle |0\rangle$$

Then we can examine the statistics by expanding the exponential

$$\left\langle \sigma(\eta_1)\sigma(\eta_2)\exp\left(i\kappa^{1/2}\int d^2z\psi(z)c^\dagger(z,\bar{z})\right)\right\rangle |0\rangle$$
$$\approx \langle\sigma(\eta_1)\sigma(\eta_2)\rangle + \frac{1}{2}\int d^2z_1 d^2z_2 \langle\sigma(\eta_1)\sigma(\eta_2)\psi(z_1)\psi(z_2)\rangle c^\dagger(z_1,\bar{z}_1)c^\dagger(z_1,\bar{z}_2)\,|0\rangle$$

and examining its short range behavior:

$$\sigma(\eta_1)\psi(z_1) \sim \frac{1}{(\eta_1 - z_1)^{1/2}}\mu(w,\bar{w})$$

Then exchanging η_1 and z_1 goes through a branch cut which modifies the phase by π. So this trial state indeed describes the phase winding of a Cooper pair around a vortex.

References

The two primary references are:

Conformal Field Theory, Philippe Francesco, Pierre Mathieu, David Senechal
Conformal Field Theory and Critical Phenomena, Malte Henkel
Some information from
Symmetries, Lie Algebras Representations: A Graduate Course for Physicists, Jorgen Fuchs
was useful for the section on Virasoro algebras and Verma modules. The section on $p_x + ip_y$ superconductors is taken from:
CFT and 2d chiral topological phases: an introduction, J. Dubail

but a similar discussion without conformal field theory, which I used for background can be found in:
Field Theories of Condensed Matter Physics, Eduardo Fradkin